Some Fun with Divergent Series

1. Preliminary Results

We begin by examining the (divergent) infinite series

$$S_1 = 1 + 2 + 3 + 4 + 5 + 6 + \cdots = \sum_{k=1}^{\infty} k$$

$$S_2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \cdots = \sum_{k=1}^{\infty} k^2$$

(i) Define $T_0 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$

Rewrite as $T_0 = \quad 1 - 1 + 1 - 1 + 1 - 1 + \cdots$

Add $2T_0 = 1 + 0 + 0 + 0 + 0 + 0 + \cdots = 1$

$$T_0 = ½$$

(ii) Define $s = 1 - 3 + 5 - 7 + 9 - 11 + \cdots$

Rewrite as $s = \quad 1 - 3 + 5 - 7 + 9 - 11 + \cdots$

Add $2s = 1 - 2 + 2 - 2 + 2 - 2 + \cdots = 1 - 2s_1 = 1 - 1$

$$s = 0$$

(iii) Define $T_1 = 1 - 2 + 3 - 4 + 5 - 6 + \cdots$

Rewrite as $T_1 = \quad 1 - 2 + 3 - 4 + 5 - 6 + \cdots$

Add $2T_1 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots = T_0 = ½$

$$T_1 = ¼$$

(iv) Define $T_2 = 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \cdots$

Rewrite as $T_2 = \quad 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \cdots$

Add $2T_2 = 1^2 - (2^2 - 1^2) + (3^2 - 2^2) - (4^2 - 3^2) + (5^2 - 4^2) - (6^2 - 5^2) + \cdots$

$= 1 - (2 - 1)(2 + 1) + (3 - 2)(3 + 2) - (4 - 3)(4 + 3) + (5 - 4)(5 + 4) - \cdots$

$= 1 - 3 + 5 - 7 + 9 - 11 + \cdots = s = 0$

$$T_2 = 0$$

Note that these results can be formalized by using summability methods, such as those attributed to Abel, Cesàro and others, which define the sum of an infinite series more generally than the usual method of determining the limit to which the partial sums of the series converge. They are designed so that values can be attached to certain series that would otherwise be considered divergent. Abel summability is described later.

2. The Series S_1 and S_2

$$S_1 - T_1 = 1 + 2 + 3 + 4 + 5 + 6 + \cdots$$

$$-1 + 2 - 3 + 4 - 5 + 6 - \cdots$$

$$= 0 + 4 + 0 + 8 + 0 + 12 + \cdots$$

$$= 4(1 + 2 + 3 + 4 + \cdots) = 4S_1$$

Hence
$$3S_1 = -T_1 = -\tfrac{1}{4}$$

$$S_1 = -\tfrac{1}{12}$$

Now consider

$$S_2 - T_2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \cdots$$

$$-1^2 + 2^2 - 3^2 + 4^2 - 5^2 + 6^2 - \cdots$$

$$= 0 + 2 \cdot (2 \cdot 1)^2 + 0 + 2 \cdot (2 \cdot 2)^2 + 0 + 2 \cdot (2 \cdot 3)^2 + \cdots$$

$$= 2^3 \cdot (1^2 + 2^2 + 3^2 + 4^2 + \cdots) = 8S_2$$

Hence
$$7S_2 = -T_2 = 0$$

$$S_2 = 0$$

3. Summary

We have 'proved' the strange and apparently nonsensical results

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots = -\tfrac{1}{12}$$

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \cdots = 0$$

In fact these results do make sense when interpreted as special values of the Riemann zeta function. The left hand sides of the equations above are no longer sums in the normal sense but rather a (misleading) notation for the zeta function evaluated at -1 and -2 respectively. We return to this later.

4. Abel Summation

An infinite series $\Sigma_{k=0}^{\infty} u_k$ is convergent to the sum S when $\lim_{n\to\infty} S_n = S$ where $S_n = \Sigma_{k=0}^{n} u_k$ is the partial sum to n terms. The series is Abel summable to S when the series $\Sigma_{k=0}^{\infty} u_k x^k$ is convergent to $S(x)$ for $0 \le x < 1$, and $\lim_{x\to 1} S(x) = S$. For example,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \quad (|x| < 1)$$

and $1/(1+x) \to \tfrac{1}{2}$ as $x \to 1$. Formally the right-hand side becomes the divergent series s_1 when $x \to 1$ so that, by definition, the Abel sum of T_0 is ½ which is the result in (i).

Note that the Abel sum of a convergent series is the same as its ordinary sum. For example, the Abel sum of the convergent series

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!}+\cdots=e$$

is defined as the limit as $x \to 1$ of

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^n}{n!}+\cdots=e^x$$

which is the value e to which the original series converges in the normal sense. Thus Abel summation is stronger than the usual method of taking the limit of the partial sums in the sense that it gives the right answer when the series is convergent but also yields a value for certain divergent series.

Further examples of Abel summation:

A binomial expansion gives

$$\frac{1}{(1+x)^2}=1-2x+3x^2-4x^3+\cdots+(-1)^n(n+1)x^n+\cdots \quad (|x|<1)$$

As $x \to 1$ the left-hand side approaches ¼ and the right-hand side takes the form of T_1. This confirms the result in (iii).

Expansion in a Maclaurin series gives

$$\frac{1-x^2}{(1+x^2)^2}=1-3x^2+5x^4-7x^6+\cdots+(-1)^n(2n+1)x^{2n}+\cdots \quad (|x|<1)$$

The left-hand side approaches 0 as $x \to 1$ and the series becomes s verifying the result in (ii).

Likewise we have

$$\frac{1-x}{(1+x)^3}=1-2^2x+3^2x^2-4^2x^3+\cdots+(-1)^n(n+1)^2x^n+\cdots \quad (|x|<1)$$

Again the left-hand side approaches 0 as $x \to 1$ while the right-hand side becomes the series T_2 which shows that the result in (iv) is also an Abel summation.

Note that the two series we started with do not have Abel sums. More sophisticated methods involving the Riemann zeta function are required to justify them.

5. The Riemann Zeta Function

For real $x > 1$, the Riemann zeta function is defined by the convergent infinite series

$$\zeta(x)=1+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\frac{1}{5^x}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^x} \qquad (1)$$

It has a singularity at $x = 1$ where it reduces to the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

which is well-known to be divergent. If we now replace the argument of the zeta function by the complex variable $z = x + iy$, then the series remains convergent provided that $x > 1$, i.e. $\zeta(z)$ is defined in the half-plane Re $z > 1$. Moreover, by defining $\zeta(z)$ in a different way which reduces to (1) for Re $z > 1$ we can extend the domain in which it is defined to cover the entire complex plane apart from the singularity at $z = 1$, a process known as analytic continuation.

Consider

$$\int_0^\infty \frac{t^{z-1}}{e^t - 1} dt = \int_0^\infty \frac{e^{-t} t^{z-1}}{1 - e^{-t}} dt = \int_0^\infty e^{-t} t^{z-1} \sum_{n=0}^\infty e^{-nt}\, dt = \sum_{n=1}^\infty \int_0^\infty t^{z-1} e^{-nt} dt$$

Substitute $nt = r$

$$\int_0^\infty \frac{t^{z-1}}{e^t - 1} dt = \sum_{n=1}^\infty \int_0^\infty (r/n)^{z-1} e^{-r} n^{-1} dr = \sum_{n=1}^\infty \int_0^\infty r^{z-1} e^{-r} dr = \zeta(z)\Gamma(z)$$

where $\Gamma(z) = \int_0^\infty r^{z-1} e^{-r} dr$ is the well-known gamma function which is analytic except at the origin and when z is a negative integer, and which reduces to $(z-1)!$ when z is a positive integer.

The advantage of this expression is that it suggests how the analytic continuation of $\zeta(z)$ should proceed. We consider the contour integral in the complex w-plane, $w = u + iv$,

$$f(z) = \int_C \frac{w^{z-1}}{e^{-w} - 1} dw = \int_C \frac{e^{(z-1)\log w}}{e^{-w} - 1} dw \qquad (2)$$

where the contour C runs from $-\infty$ along the upper side of the branch cut $u < 0$, $v = 0$ of the logarithmic function, encloses the origin in a circle of vanishingly small radius, and then returns to $-\infty$ along the lower side of the branch cut. Note that $f(z)$ is analytic for all finite values of z because the contour integral is uniformly convergent in any finite region of the complex z-plane. On the upper side of the branch cut we have $\log w = \log|u| + i\pi$, and on the lower side $\log w = \log|u| - i\pi$. It is easily shown that if $x = \text{Re}\, z > 1$ the contribution to the integral around the origin vanishes as the radius of the circle tends to zero. Thus

$$f(z) = \int_{-\infty}^0 \frac{e^{(z-1)(\log|u|+i\pi)}}{e^{-u} - 1} du + \int_0^{-\infty} \frac{e^{(z-1)(\log|u|-i\pi)}}{e^{-u} - 1} du = 2i \sin z\pi \int_0^{-\infty} \frac{(-u)^{z-1}}{e^{-u} - 1} du$$

$$= -2i \sin z\pi \int_0^\infty \frac{t^{z-1}}{e^t - 1} du = -\,2i \sin z\pi\, \Gamma(z)\zeta(z) \qquad (\text{Re}\, z > 1)$$

The last step follows from (2) but the restriction on z can be relaxed in this formula because we already know that $f(z)$ is analytic in the entire finite region of the complex z–plane. Thus with the aid of a well-known identity, $\Gamma(z)\Gamma(1-z) = \pi \csc z\pi$, we obtain

$$\zeta(z) = \frac{i\Gamma(1-z)}{2\pi}\int_C \frac{w^{z-1}}{e^{-w}-1}dw \tag{3}$$

We now determine in which regions of the complex z-plane, in addition to Re $z > 1$, this formula defines $\zeta(z)$ as an analytic function. We know already that the integral is analytic in the entire finite plane so the only possible singularities of $\zeta(z)$ are those of $\Gamma(1-z)$, namely simple poles at $z = 1, 2, 3, \ldots$ all of which except $z = 1$ can be discarded because we already know that $\zeta(z)$ is analytic for Re $z > 1$. Thus (3) represents an analytic continuation of $\zeta(z)$ to all finite points in the complex z–plane except $z = 1$ where it has a simple pole.

When z is an integer, the integrals along C from $-\infty$ to the origin and back cancel each out because there is no need to construct a branch cut to accommodate a logarithmic function in this case (w^{m-1}, m an integer, is analytic everywhere). If one did continue to write $e^{(m-1)\log w}$ for w^{m-1}, the resulting factor $\sin(m-1)\pi$ in the combined integrals would in any case ensure the vanishing of the total integral along that part of C. Thus it is only the vanishingly small circle around the origin that will contribute to the contour integral when $z = m$, an integer. We have already seen that when Re $z > 1$ (i.e. $m = 2, 3, 4 \ldots$) this contribution is zero, which is as it should be in order to nullify the singularities in $\Gamma(1-m)$ at those points and thereby ensuring the analyticity of $\zeta(z)$ for Re $z > 1$. We are left to consider the values $m = 0, -1, -2, -3 \ldots$ (we showed above that $m = 1$ is a singularity – a simple pole). Let the circular part of C around the origin have radius ϵ, let the integrand in (3) be expanded in a (Laurent) series of powers of w, and consider the integration around the origin of one such term $a_p w^p$ where p is an integer (positive, negative or zero) and a_p is a coefficient. With $w = \epsilon e^{i\theta}$ we obtain

$$\int_C a_p w^p dw = \int_{\pi}^{-\pi} i a_p \epsilon^{p+1} e^{i(p+1)\theta} d\theta = -2ia_p\epsilon^{p+1}\frac{\sin(p+1)\pi}{p+1} \to \begin{cases} 0 & (p \neq -1) \\ -2\pi i a_p & (p = -1) \end{cases}$$

as $\epsilon \to 0$. Thus only terms of type w^{-1} contribute to the contour integral, a property that will be familiar from the calculus of residues.

With $z = m$, the integrand in (3) can be expanded in terms of the Bernoulli numbers B_n defined by the generating function $t/(e^t - 1) = \Sigma_{n=0}^{\infty}(B_n t^n/n!)$. The first values are $B_0 = 1, B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}$. Thus for $0 < |w| < 2\pi$ (to avoid the poles at $w = \pm 2\pi i$), we have

$$\frac{w^{m-1}}{e^{-w}-1} = -w^{m-2}\frac{(-w)}{e^{-w}-1} = -w^{m-2}\sum_{n=0}^{\infty} B_n \frac{(-w)^n}{n!} = -w^{m-2}\left(1 + \frac{w}{2} + \frac{w^2}{12} - \frac{w^4}{720} + \cdots\right)$$

For $m = 0$ the contributing term is the second one with coefficient $-½$. Thus the integral in (3) is evaluated as πi whence $\zeta(0) = -½\Gamma(1) = -½$. For $m = -1$ the relevant term is the third with coefficient $-1/12$, the integral has the value $\pi i/6$, and $\zeta(-1) = -\Gamma(2)/12 = -1/12$. There is no term of the form w^{-1} when $m = -2$ so its coefficient is 0 and the integral is also zero. Thus $\zeta(-2) = 0$. Likewise, for $m = -3$ we deduce from the fourth term in the series that $\zeta(-3) = \Gamma(4)/720 = 3!/720 = 1/120$.

If we formally substitute these results in (1), the original definition of $\zeta(x)$ for real $x > 1$, we obtain

$$\zeta(0) = 1 + 1 + 1 + 1 + 1 + 1 + \cdots = -\tfrac{1}{2}$$

$$\zeta(-1) = 1 + 2 + 3 + 4 + 5 + 6 + \cdots = -\tfrac{1}{12}$$

$$\zeta(-2) = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \cdots = 0$$

$$\zeta(-3) = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + \cdots = \tfrac{1}{120}$$

The second and third of these series will be recognized as the sums S_1 and S_2 with which we started this discussion. Amazingly, however, the apparently absurd results we found by boldly adventurous but totally invalid manipulations of divergent series have been confirmed rigorously as the relevant values of the Riemann zeta function. They cannot be interpreted as sums of series in the usual sense, of course. The series must be regarded as alternative notations for values of the zeta function.

The first and fourth results above suggest we should be able to "prove" by similar questionable methods that the respective series, which we can denote by S_0 and S_3, have sums of $-½$ and 1/120. The first result is easy to show by following the procedure we first used to derive S_1 and S_2. Thus

$$\begin{aligned} S_0 - T_0 &= 1 + 1 + 1 + 1 + 1 + 1 + \cdots \\ &\quad -1 + 1 - 1 + 1 - 1 + 1 - \cdots \\ &= 0 + 2 + 0 + 2 + 0 + 2 + \cdots \\ &= 2(1 + 1 + 1 + 1 + 1 + 1 + \cdots) = 2S_0 \end{aligned}$$

Hence $$S_0 = -T_0 = -½$$

To find the sum S_3 we use exactly the same recipe in a more general notation. Let $S_3 = \Sigma_{k=1}^{\infty} k^3$ and $T_3 = \Sigma_{k=1}^{\infty} (-1)^{k+1} k^3$ or alternatively $T_3 = \Sigma_{k=2}^{\infty} (-1)^k (k-1)^3$. Adding the last two expressions, we have

$$\begin{aligned} 2T_3 &= 1 + \Sigma_{k=2}^{\infty} (-1)^{k+1} [k^3 - (k-1)^3] = 1 + \Sigma_{k=2}^{\infty} (-1)^{k+1} [1 + 3k(k-1)] \\ &= \Sigma_{k=1}^{\infty} (-1)^{k+1} [1 - 3k(k+1)] = T_0 - 3T_2 - 3T_1 = \tfrac{1}{2} - 0 - \tfrac{3}{4} = -\tfrac{1}{4} \end{aligned}$$

i.e. $T_3 = -1/8$. Now

$$S_3 - T_3 = \Sigma_{k=1}^{\infty} [k^3 - (-1)^{k+1} k^3] = \Sigma_{k=1}^{\infty} 2(2k)^3 = 16\Sigma_{k=1}^{\infty} k^3 = 16S_3$$

so that $15S_3 = -T_3 = 1/8$. Thus $S_3 = 1/120$.

6. The General Case

Let $T_n = \Sigma_{k=1}^{\infty}(-1)^{k+1}k^n$, $(n = 0, 1, 2, \dots)$. An alternative but equivalent way of writing this expression is $T_n = \Sigma_{k=2}^{\infty}(-1)^k(k-1)^n$. Adding the two together we have

$$2T_n = 1 + \Sigma_{k=2}^{\infty}(-1)^{k+1}[k^n - (k-1)^n]$$

yielding $T_0 = ½$, and for $n \geq 1$,

$$\begin{aligned} 2T_n &= 1 - \Sigma_{k=2}^{\infty}(-1)^{k+1}\Sigma_{m=0}^{n-1}(-1)^{n+m}\binom{n}{m}k^m \\ &= 1 - \Sigma_{k=1}^{\infty}(-1)^{k+1}\Sigma_{m=0}^{n-1}(-1)^{n+m}\binom{n}{m}k^m + \Sigma_{m=0}^{n-1}(-1)^{n+m}\binom{n}{m} \\ &= -\Sigma_{k=1}^{\infty}(-1)^{k+1}\Sigma_{m=0}^{n-1}(-1)^{n+m}\binom{n}{m}k^m \\ &= \Sigma_{m=0}^{n-1}(-1)^{n+m+1}\binom{n}{m}\Sigma_{k=1}^{\infty}(-1)^{k+1}k^m \\ &= \Sigma_{m=0}^{n-1}(-1)^{n+m+1}\binom{n}{m}T_m \end{aligned}$$

Together with the initial value of $T_0 = ½$ from above, this is the recursion relation for T_n. It can be expressed more tidily by defining $V_n = (-1)^n T_n$ so that

$$V_0 = ½, \quad 2V_n = -\Sigma_{m=0}^{n-1}\binom{n}{m}V_m \quad (n \geq 1) \tag{4}$$

The binomial expansion $(k-1)^n = (-1)^n(1-k)^n = \Sigma_{m=0}^{n}(-1)^{n+m}\binom{n}{m}k^m$ has been used in the derivation above, with $\binom{n}{m} = n!/[(n-m)!\,m!]$ in the usual notation. We also noted the fact that when $k = 1$ this expansion gives $0 = \Sigma_{m=0}^{n}(-1)^{n+m}\binom{n}{m} = 1 + \Sigma_{m=0}^{n-1}(-1)^{n+m}\binom{n}{m}$.

The first few values of T_n generated by the recurrence relation are

$$T_1 = \frac{1}{4}$$

$$T_2 = -\frac{1}{4} + \frac{1}{2}\binom{2}{1}T_1 = -\frac{1}{4} + \frac{1}{4} = 0$$

$$T_3 = \frac{1}{4} - \frac{1}{2}\binom{3}{1}T_1 + \frac{1}{2}\binom{3}{2}T_2 = \frac{1}{4} - \frac{3}{8} + 0 = -\frac{1}{8}$$

$$T_4 = -\frac{1}{4} + \frac{1}{2}\binom{4}{1}T_1 - \frac{1}{2}\binom{4}{2}T_2 + \frac{1}{2}\binom{4}{3}T_3 = -\frac{1}{4} + \frac{1}{2} - 0 - \frac{1}{4} = 0$$

$$T_5 = \frac{1}{4} - \frac{1}{2}\binom{5}{1}T_1 + 0 - \frac{1}{2}\binom{5}{3}T_3 + 0 = \frac{1}{4} - \frac{5}{8} + \frac{5}{8} = \frac{1}{4}$$

$$T_6 = -\frac{1}{4} + \frac{1}{2}\binom{6}{1}T_1 - 0 + \frac{1}{2}\binom{6}{3}T_3 - 0 + \frac{1}{2}\binom{6}{5}T_5 = -\frac{1}{4} + \frac{3}{4} - \frac{5}{4} + \frac{3}{4} = 0$$

$$T_7 = \frac{1}{4} - \frac{1}{2}\binom{7}{1}T_1 + 0 - \frac{1}{2}\binom{7}{3}T_3 + 0 - \frac{1}{2}\binom{7}{5}T_5 + 0 = \frac{1}{4} - \frac{7}{8} + \frac{35}{16} - \frac{21}{8} = -\frac{17}{16}$$

$$T_8 = -\frac{1}{4} + \frac{1}{2}\binom{8}{1}T_1 - 0 + \frac{1}{2}\binom{8}{3}T_3 - 0 + \frac{1}{2}\binom{8}{5}T_5 - 0 + \frac{1}{2}\binom{8}{7}T_7 = -\frac{1}{4} + 1 - \frac{7}{2} + 7 - \frac{17}{4} = 0$$

Now let $S_n = \Sigma_{k=1}^{\infty} k^n = 1^n + 2^n + 3^n + 4^n + 5^n + 6^n + \cdots$. Then

$$S_n - T_n = \Sigma_{k=1}^{\infty}[1-(-1)^{k+1}]k^n = \Sigma_{k=2,4,6,\ldots}^{\infty} 2k^n = \Sigma_{k=1}^{\infty} 2(2k)^n = 2^{n+1}\Sigma_{k=1}^{\infty} k^n = 2^{n+1}S_n$$

Thus
$$S_n = -T_n/(2^{n+1}-1) = (-1)^{n+1}V_n/(2^{n+1}-1) \qquad (5)$$

and from the previous results for T_n

$$1+1+1+1+1+1+\cdots = S_0 = -T_0 = -\tfrac{1}{2}$$

$$1+2+3+4+5+6+\cdots = S_1 = -\tfrac{1}{3}T_1 = -\tfrac{1}{12}$$

$$1^2+2^2+3^2+4^2+5^2+6^2+\cdots = S_2 = -\tfrac{1}{7}T_2 = 0$$

$$1^3+2^3+3^3+4^3+5^3+6^3+\cdots = S_3 = -\tfrac{1}{15}T_3 = \tfrac{1}{120}$$

$$1^4+2^4+3^4+4^4+5^4+6^4+\cdots = S_4 = 0$$

$$1^5+2^5+3^5+4^5+5^5+6^5+\cdots = S_5 = -\tfrac{1}{63}T_5 = -\tfrac{1}{252}$$

$$1^6+2^6+3^6+4^6+5^6+6^6+\cdots = S_6 = 0$$

$$1^7+2^7+3^7+4^7+5^7+6^7+\cdots = S_7 = -\tfrac{1}{255}T_7 = \tfrac{1}{240}$$

$$1^8+2^8+3^8+4^8+5^8+6^8+\cdots = S_8 = 0$$

It seems highly probable from these calculations that T_n vanishes when n is even (likewise with V_n and S_n by definition). To prove this consider the generating function

$$g(t) = \frac{1}{e^t+1} = \sum_{m=0}^{\infty} a_m \frac{t^m}{m!}$$

where the coefficients a_m are to be determined. Defining $b_j = 1\ (j \neq 0), b_0 = 2,$ so that we can write $e^t + 1 = \Sigma_{j=0}^{\infty} b_j t^j/j!$, we deduce from the equation above

$$1 = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} b_j a_m \frac{t^{j+m}}{j!\,m!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{b_{n-m}a_m t^n}{m!\,(n-m)!} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{m=0}^{n} b_{n-m}a_m \binom{n}{m}$$

The second form of the double summation above is found by noting that all the points (j,m) in the first quadrant (covered by the first double summation) can be counted alternatively along each sloping line defined by $j+m=n$ and then summing over all such lines from $n=0$ to ∞. As m runs through its values from 0 to n along each line defined by a particular value of n, the value of the index j is defined by $j = n-m$ which has been substituted in the second double summation.

Since $b_{n-m} = 2$ when $m = n$ and 1 otherwise, we may rewrite the last equation as

$$1 = 2a_0 + \sum_{n=1}^{\infty} \frac{t^n}{n!}\left[2a_n + \sum_{m=0}^{n-1} a_m \binom{n}{m}\right]$$

Equating coefficients of the powers of t we obtain

$$a_0 = ½, \quad 2a_n + \sum_{m=0}^{n-1} a_m \binom{n}{m} = 0 \quad (n \geq 1)$$

This is the same recurrence relation as (4) showing that $a_n = V_n$ and hence that $g(t)$ is the generating function for $V_n = (-1)^n T_n$, i.e.

$$g(t) = \sum_{m=0}^{\infty} V_m \frac{t^m}{m!} = \frac{1}{2} + \sum_{m=1}^{\infty} V_m \frac{t^m}{m!} \tag{6}$$

Define $f(t) = g(t) - ½ = \Sigma_{m=1}^{\infty} V_m t^m/m!$. Then

$$f(t) = \frac{1}{e^t + 1} - \frac{1}{2} = \frac{1 - e^t}{2(1 + e^t)} = \frac{e^{-t} - 1}{2(e^{-t} + 1)} = -f(-t)$$

Thus $f(t)$ is an odd function so that all coefficients of even powers of t in the expansion of $f(t)$ above must vanish, i.e. $V_{2n} = 0$ $(n \geq 1)$. It follows that T_{2n} and S_{2n} also vanish for $n \geq 1$. Moreover, since only the odd-indexed terms V_n are non-vanishing for $n \geq 1$, we see from (5) that

$$V_0 = ½, \; V_n = (2^{n+1} - 1)S_n \; (n \geq 1) \tag{7}$$

since $(-1)^{n+1} = 1$ when n is odd. It follows from the definition of g that

$$tg(t) = \frac{t}{2} + \sum_{m=1}^{\infty} (2^{m+1} - 1) S_m \frac{t^{m+1}}{m!} = \frac{t}{2} + \sum_{n=2}^{\infty} S_{n-1} \frac{(2t)^n - t^n}{(n-1)!}$$

We now define $B_n = -nS_{n-1}$ $(n \geq 2)$, $B_1 = -½$ with B_0 undefined for the time being. Expressed in terms of B_n the equation above becomes

$$tg(t) = \sum_{n=1}^{\infty} B_n \left[\frac{t^n}{n!} - \frac{(2t)^n}{n!}\right] = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} - \sum_{n=0}^{\infty} B_n \frac{(2t)^n}{n!}$$

The change in the lower limit of the first summation is permissible because of the way B_1 has been defined and in the second summation because B_0 cancels out in the subtraction of the two series. If $h(t) = \Sigma_{n=0}^{\infty} B_n t^n/n!$ is the generating function for B_n, the last equation can be expressed in terms of this generating function by $tg(t) = h(t) - h(2t)$. Now

$$tg(t) = \frac{t}{e^t + 1} = \frac{t(e^t - 1)}{e^{2t} - 1} = \frac{t(e^t + 1) - 2t}{e^{2t} - 1} = \frac{t}{e^t - 1} - \frac{2t}{e^{2t} - 1}$$

which shows that $h(t) = t/(e^t - 1)$. Clearly $h(0) = 1$ thereby fixing the value of B_0 as 1. Now $h(t)$ is the generating function for the well-known Bernoulli numbers introduced in Section 5 in connection the Riemann zeta function. Thus the quantities B_n defined here are actually the same familiar Bernoulli numbers. Thus with the aid of (5) and (7) we can now express the original recursive quantities T_n in terms of Bernoulli numbers by the formula

$$T_0 = -B_1 = ½, \quad T_n = (2^{n+1} - 1)B_{n+1}/(n + 1) \quad (n \geq 1).$$

Likewise, by (5) the sums of powers of the natural numbers series by

$$S_0 = B_1, \quad S_n = -B_{n+1}/(n + 1) \quad (n \geq 1).$$

7. The Generating Function $g(t)$

In the last section I inferred the form of the generating function (6) indirectly and then verified that it did indeed yield the correct recursion relation for V_n. After completing this article, however, I realised I could derive the generating function directly by adapting a clever technique used by Tao[1] to obtain the generating function for the Bernoulli numbers. We rewrite (4) as

$$\sum_{m=0}^{n-1} \frac{n!}{(n-m)!}\frac{V_m}{m!} + 2n!\frac{V_n}{n!} = 0 \quad (n \geq 1) \tag{8}$$

and note that with $D_m \equiv d^m/dx^m$

$$D_m x^n = \begin{cases} \dfrac{n!\,x^{n-m}}{(n-m)!} & (m \leq n) \\ 0 & (m > n) \end{cases}$$

It follows that

$$[D_m x^n]_{x=1} + [D_m x^n]_{x=0} = \begin{cases} n!/(n-m)! & (m < n) \\ 2n! & (m = n) \\ 0 & (m > n) \end{cases}$$

so that (8) can be expressed in the form

$$\sum_{m=0}^{\infty} \frac{V_m}{m!}\{[D_m x^n]_{x=1} + [D_m x^n]_{x=0}\} = 0 \qquad (n \geq 1).$$

When $n = 0$ the left-hand side is $2V_0 = 1$ since $V_0 = ½$. Thus the case $n = 0$ can be incorporated in the equation by writing it in the form

$$\sum_{m=0}^{\infty} \frac{V_m}{m!}\{[D_m x^n]_{x=1} + [D_m x^n]_{x=0}\} = [x^n]_{x=0} \qquad (n \geq 0).$$

We can build on this result through linearity. Thus

$$\sum_{m=0}^{\infty} \frac{V_m}{m!}\{[D_m(a_n x^n + a_k x^k)]_{x=1} + [D_m(a_n x^n + a_k x^k)]_{x=0}\} = [a_n x^n + a_k x^k]_{x=0}$$

and in general, if $P^{(m)}(x)$ denotes the m^{th} derivative of the polynomial $P(x) = \Sigma_{m=0}^{n} a_m x^m$, then

$$\sum_{m=0}^{\infty} \frac{V_m}{m!} \left\{P^{(m)}(1) + P^{(m)}(0)\right\} = P(0). \qquad (9)$$

This result can be extended to include convergent power series of the form $P(x) = \Sigma_{m=0}^{\infty} a_m x^m$.

Let $a_m = t^m/m!$. Then we have $P(x) = \Sigma_{m=0}^{\infty} (tx)^m/m! = e^{tx}$, with values $P^{(m)}(1) = t^m e^t$, $P^{(m)}(0) = t^m$ and $P(0) = 1$. Thus from (9)

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} V_m \{e^t + 1\} = 1$$

which yields the generating function

$$g(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} V_m = \frac{1}{e^t + 1}$$

in agreement with (6).

[1] https://terrytao.wordpress.com/2010/04/10/the-euler-maclaurin-formula-bernoulli-numbers-the-zeta-function-and-real-variable-analytic-continuation/