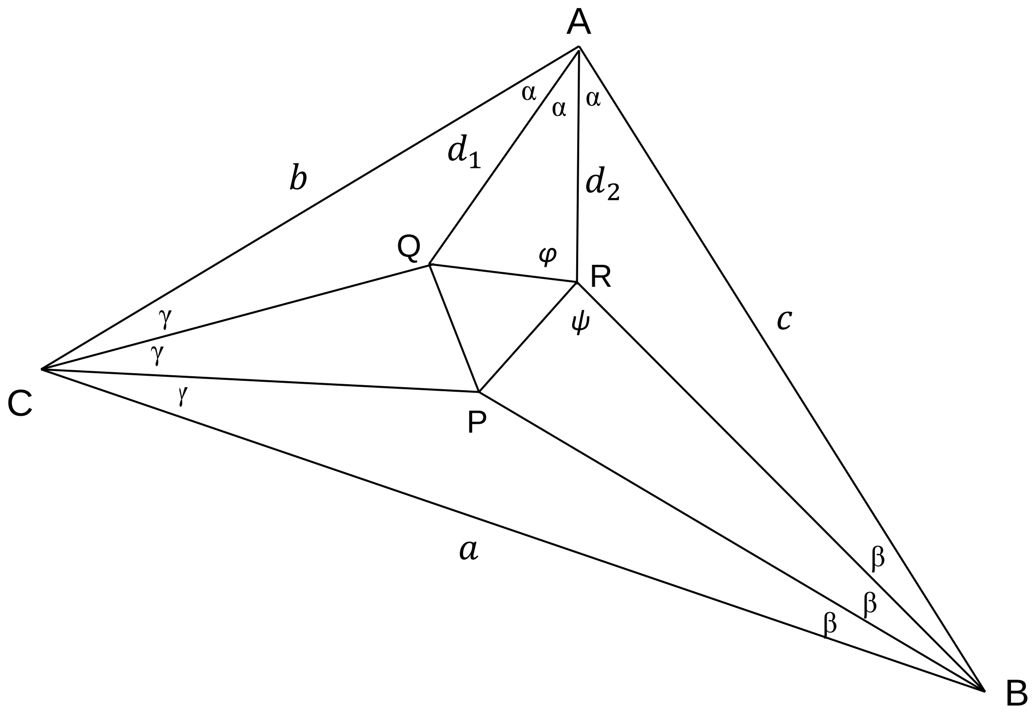
Morley's 'Miracle'



Morley's theorem states that the intersections of the trisectors of the angles of a triangle form the vertices of an equilateral triangle. Let $A = 3\alpha, B = 3\beta$ and $C = 3\gamma$ as shown in the figure, so that with angles measured in degrees

$$\alpha + \beta + \gamma = 60. \tag{1}$$

Thus $\angle \text{ARB} = 180 - \alpha - \beta = 120 + \gamma$. Similarly $\angle \text{AQC} = 120 + \beta$. Let $\angle \text{ARQ} = \varphi$, $\angle \text{BRP} = \psi$ so that

$$\angle \text{QRP} = 360 - (120 + \gamma) - \varphi - \psi = 240 - \gamma - \varphi - \psi. \tag{2}$$

Our aim is to prove $\angle \text{QRP} = 60$. First we derive two trigonometric results.

(i) For any angle θ

$$\sin 3\theta = \sin(\theta + 2\theta) = \sin\theta\cos 2\theta + \cos\theta\sin\theta$$

$$= \sin\theta\,(4\cos^2\theta - 1) = 4\sin\theta\left(\cos\theta - \tfrac{1}{2}\right)\left(\cos\theta + \tfrac{1}{2}\right)$$

$$= 4\sin\theta\,(\cos\theta - \cos 60)(\cos\theta + \cos 60)$$

$$= 16\sin\theta\sin\frac{60+\theta}{2}\sin\frac{60-\theta}{2}\cos\frac{60+\theta}{2}\cos\frac{60-\theta}{2}$$

$$= 4\sin\theta\sin(60+\theta)\sin(60-\theta). \tag{3}$$

(ii) For the angles α, β and γ in the triangle ABC

$$\sin(60+\gamma) = \sin(120-\alpha-\beta) \qquad \text{by (1)}$$

$$= \sin(180-[120-\alpha-\beta])$$

$$= \sin(60+\alpha+\beta)$$

$$= \sin(60+\beta)\cos\alpha + \sin\alpha\cos(60+\beta). \tag{4}$$

The proof now follows through straightforward applications of the Sine Law which for the triangle ABC states that

$$\frac{a}{\sin 3\alpha} = \frac{b}{\sin 3\beta} = \frac{c}{\sin 3\gamma} = 2R$$

where R is the radius of the triangle's circumscribed circle. In triangles AQR, ARB and AQC it yields

$$\frac{d_1}{\sin\varphi} = \frac{d_2}{\sin(180-\alpha-\varphi)} = \frac{d_2}{\sin(\alpha+\varphi)}$$

$$\frac{d_2}{\sin\beta} = \frac{c}{\sin(120+\gamma)} = \frac{c}{\sin(60-\gamma)} \qquad \frac{d_1}{\sin\gamma} = \frac{b}{\sin(60-\beta)}$$

Eliminating d_1 and d_2 and substituting for b and c we obtain

$$\frac{2R\sin 3\beta\sin\gamma}{\sin(60-\beta)\sin\varphi} = \frac{2R\sin 3\gamma\sin\beta}{\sin(60-\gamma)\sin(\alpha+\varphi)}$$

After further simplification, with the aid of (3), this equation reduces to

$$[\sin(60+\gamma) - \sin(60+\beta)\cos\alpha]\sin\varphi = \sin(60+\beta)\sin\alpha\cos\varphi$$

whence substitution of (4) gives

$$\tan\varphi = \tan(60+\beta) \implies \varphi = 60+\beta.$$

It follows by analogy that $\psi = 60+\alpha$, so that substitution in (2) gives

$$\angle\text{QRP} = 240-\gamma-(60+\beta)-(60+\alpha) = 120-(\alpha+\beta+\gamma) = 60$$

with the aid of (1). Since the vertex R of triangle PQR was chosen arbitrarily, the angles at the vertices P and Q will also be 60°. Thus the triangle PQR is equilateral. QED